On a Fractal Model for Turbulence

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An interpretation of a recently proposed generic model for fractal basin boundaries in terms of a cascade-like model for turbulence is presented. Intermittency effects are shown by the analysis of structure functions and non-Gaussian probability density functions.

1. Introduction

In characterizing fully developed turbulence by means of statistical methods, it is still an actual problem to explain what happens if one transits from large scales (scale of the system) to small scales (dissipation-range scale or Kolmogorov microscale). From experiments it is known that the shape of the probability density functions (PDF) of velocity differences between two points $\delta u_r = (v(y+r) - v(y))$ changes sensitively with the chosen length scale (r denotes the distance between two points; y denotes the position; v usually stands for the velocity component in direction of r). Gaussian shapes of the PDF are found for large length scales. Coming to smaller length scales, small and large velocity differences become more probable compared to a Gaussian distribution. This effect is called intermittency.

There are two common methods to characterize this intermittency effect. One may evaluate directly the changing form of the PDF. Here a set of theoretical PDFs has to be used. The question is, how well are the experimental PDFs fitted by the theoretical PDFs. An other common method is to calculate the structure functions $S^q(r) := \langle(\delta u_r)^q\rangle$, where $\langle\ldots\rangle$ denotes the spatial average. Using the structure functions to evaluate the intermittency in turbulence, one has to be very cautious because the calculations become precarious already for relatively small values of q. This point will be discussed later in this paper. To see the connection between these two methods, one should note that $S^q(r)$ corresponds to the q-th moment of the probability density function for the chosen r, denoted in the following by $P_r(\delta u_r)$:

$$S^q(r) = \int_{-\infty}^{+\infty} (\delta u_r)^q \, P_r(\delta u_r) \, \mathrm{d}\delta u_r \,. \qquad (1)$$

For the characterization of intermittency by the structure functions, scaling exponents $\zeta(q)$ are estimated under the scaling hypotheses

$$S^q(r) \sim r^{\zeta(q)} \,. \qquad (2)$$

It is easy to verify that a nonlinear function $\zeta(q)$ is directly correlated with a changing shape of the PDF with r. Further details to the problem of intermittency in turbulence can be found for example in [1–4].

This paper is devoted to the question whether intermittency effects can also be found in a fractal model [5] originally proposed by Rössler to describe fractal basin boundaries in a generic way. This question gains in significance by the fact that this fractal model is based only on a purely deterministic chaotic dynamics. Furthermore, a time invertible version [5] and a version continuous in time in form of a ODE [6] are known for this fractal set. The structure of this paper is the following. First, the basic ideas of the fractal model are repeated. The relation of the explicit analytic expression of the basin boundary to a Weierstrass function is shown. The comparison with the Weierstrass function leads naturally to a cascade interpretation. Next, the structure functions of the fractal boundary curve are evaluated. At last, the changing shape of PDFs are presented. The change from a nearly Gaussian shape to a highly intermittent shape is quantified by theoretical lognormal PDFs.

2. The Fractal Model

Recently a generic model for fractal basin boundaries was presented [5]. The main idea was that a simple

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mechanism of a chaotic forcing of a bistability leads to all kinds of different fractal curves. The schematic form of the equation is

$$x_{n+1} = F(x_n) + b\,y_n, \quad y_{n+1} = G(y_n), \tag{3}$$

where x and y are two real variables, n is the integer iteration step number and F and G are two functions. The x dynamics with F should be bistable, at $x=1$, for example. G causes a chaotic dynamics in y [for the numeric results in this paper we used $F(x) = x^\alpha$, with $\alpha=1.95$, $G(y)=3.95\,y(1-y)$, and $b=0.1$]. For $x>0$ the dynamics of (3) has two attractors, one close to zero, the other at infinity. If for any n x becomes larger than 1, the dynamics will go to infinity. From this an explicit analytic expression of the fractal boundary between these two attractors has been derived:

$$x_0(y_0) = \lim_{n\to\infty} F^{-1}(F^{-1}(\ldots F^{-1}(1-b\,G^{(n-1)}) - b\,G^{(n-2)})\ldots - b\,G^{(1)}) - b\,G^{(0)}) \tag{4}$$

with the conventions $G^{(n)} = G^{(n)}(y_0)$ and $G^{(0)} = y_0$; $G^{(n)}$ means the n-th iterate of the function G. In Fig. 1 an example of this fractal basin boundary is shown. That this fractal curve is closely related to a Weierstrass function

$$W(t) = \sum_{n=1}^{\infty} b^n \sin a^n t$$

is best seen if one regards the components $x_0^{(n)}(y_0) - x_0^{(n-1)}(y_0)$ (Figure 2). In this way the fractal curve of (4) is the sum over all such components:

$$x_0(y_0) = \sum_{n=0}^{\infty} \left(x_0^{(n)}(y_0) - x_0^{(n-1)}(y_0)\right). \tag{5}$$

Thus the similarity between a Weierstrass function and our fractal curve is shown, but one should note that the simple relation between two components of the Weierstrass function, namely

$$W^n(t) := b^n \sin(a^n t) = b^n\left(b^{n-1} \sin a^{n-1}(a\,t)\right) = b\,W^{n-1}(a\,t), \tag{6}$$

does not hold for the (5). For our fractal curve it is only by some averaging possible to show such a relation [7]. This point may be closely connected with the fact that the fractal curve has multifractal scaling behavior. Furthermore, it can be said that two components, $n-1$ and n, differ by one further application of G and F^{-1}. The meaning of G is a folding and stretching, whereas F^{-1} causes a scaling of the size of the fluctuations. As can be seen from Fig. 2, the successive

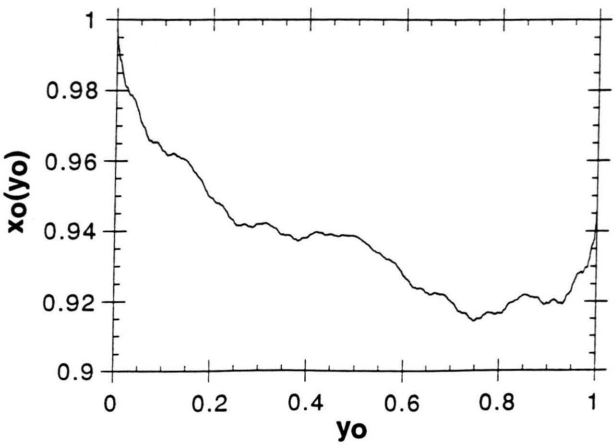

Fig. 1. Basin boundary of (3) or graph of the function (4) evaluated for $n=30$.

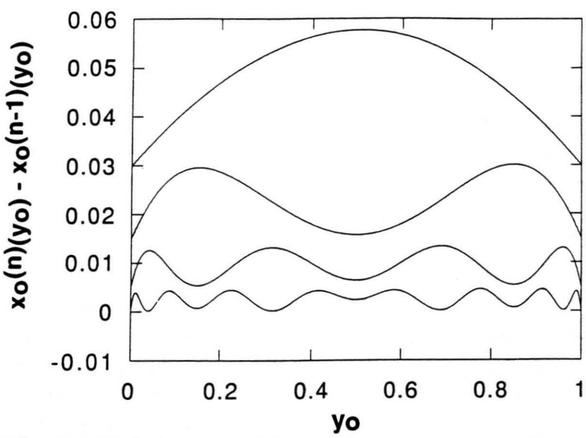

Fig. 2. Added structure of the n-th iterate, constructed from the differences of $x_0^{(n)}(y_0)$ and $x_0^{(n-1)}(y_0)$ of (4). The components for $n = 2$, 3, 4, and 5 are shown. For clarity, the curves are shifted so that they do not intersect.

application of G and F^{-1} in (4) leads to finer and finer structures which become smaller in amplitude and faster oscillating. In this way the Weierstrass-like function of (4) corresponds to the result of a successive stretching and folding mechanism.

It is often argued that a successive stretching and folding leads to the turbulent pattern in a flow. To make this idea more concrete, let us take an initially laminar flow which becomes turbulent with propagation, a jet flow for example. For this flow, we consider a spatial part, which is now our y_0 direction. Let us suppose that $x_0^{(n)}(y_0)$ denotes some quantity, like a passive scalar or the velocity (n denotes the time). The curves of Fig. 2 correspond to fluctuations developing in time. First one maximum is generated and then successively further maxima and minima are added in smaller scales, until finally a fractal curve of this quantity is obtained in space. It is easy to imagine that the maxima and minima are correlated to vortices in the flow. In this way, the curves of Fig. 2 may be compared with the Richardson cascade of turbulence, where larger vortices produce smaller and smaller vortices. One essential point of our model is that it represents a deterministic self-structuring process. Here only the different initial conditions in y_0 are essential, as it was already pointed out in [8].

3. Structure Functions and Probability Density Functions

For a turbulent flow it is well known that it corresponds to a $1/f^{\alpha}$ noise. Of actual interest are nowadays deviations from simple scaling behavior proposed by Kolmogorov [9]. This is most commonly investigated by the analysis of structure functions. For a quantity $\theta(y)$ in space y and a chosen length r the scaling of the structure function is investigated as follows:

$$S^q(r) = \langle |\theta(y+r) - \theta(y)|^q \rangle \sim r^{\zeta(q)} . \qquad (7)$$

(Note that the exponent $\zeta(2)$ is directly related to exponent α of a power spectrum.) The fractal curve of (4) was calculated for 100 000 points, and then the structure function was evaluated for different q (Figure 3). As shown in Fig. 4a, we find for the evaluation of the higher order structure functions a nonlinear dependence of the scaling exponent $\zeta(q)$ with q. This behavior of $\zeta(q)$ indicates multifractality as well as the changing shape of the underlying PDFs. The multifractality is often also evaluated by the generalized Hölder exponent [10], defined as $\zeta(q)/q$ (Figure 4b). Definitely, it has to be pointed out that the obtained scaling exponents do not represent a realistic simulation of velocity distributions in turbulence. For this one should have scaling exponents with values close to $q/3$ for small values of q. To achieve this, an analog to the conservation law of the energy transfer should be shown for our fractal model ansatz by well chosen control parameters.

As a next point the statistics of the quantities $\delta\theta(r) = \theta(y+r) - \theta(y)$ are presented. In Fig. 5 a sequence of probability density functions of our fractal curve are shown. It is clearly seen that the form of the PDFs

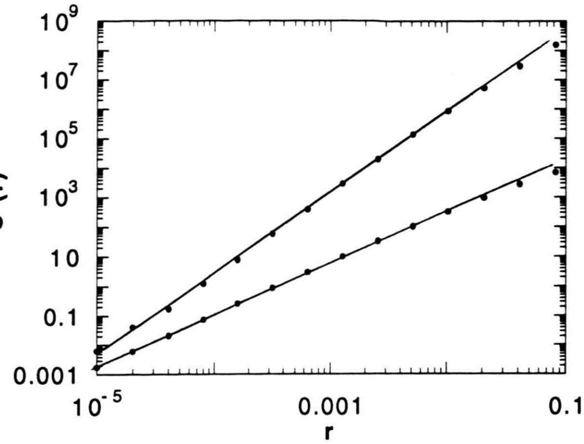

Fig. 3. Scaling of the structure function for $q=2$ and $q=4$. The straight lines correspond to exponents $\zeta(2)=1.77$ and $\zeta(4)=2.70$.

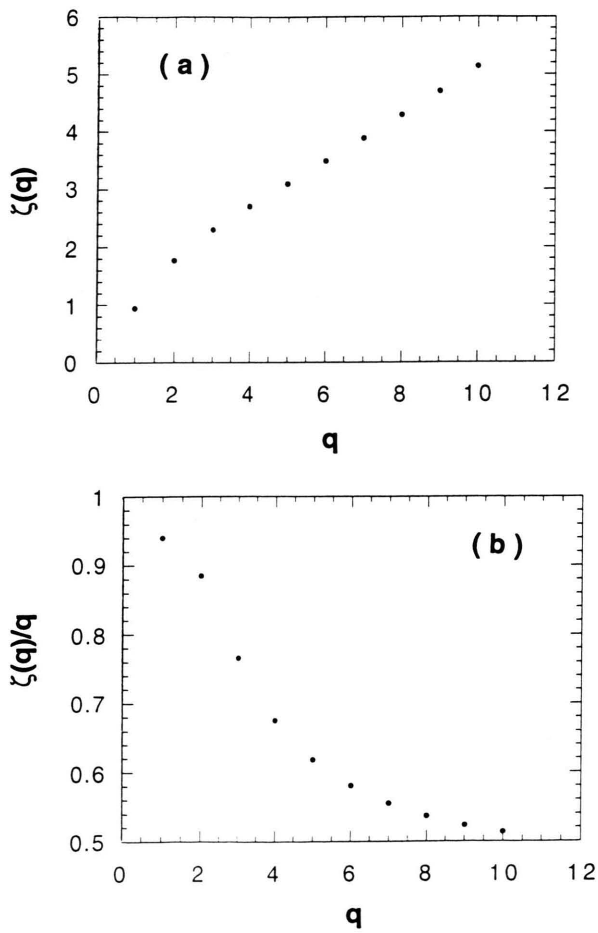

Fig. 4. (a) Structure function exponents $\zeta(q)$ versus q. (b) $\zeta(q)/q$ as function of q.

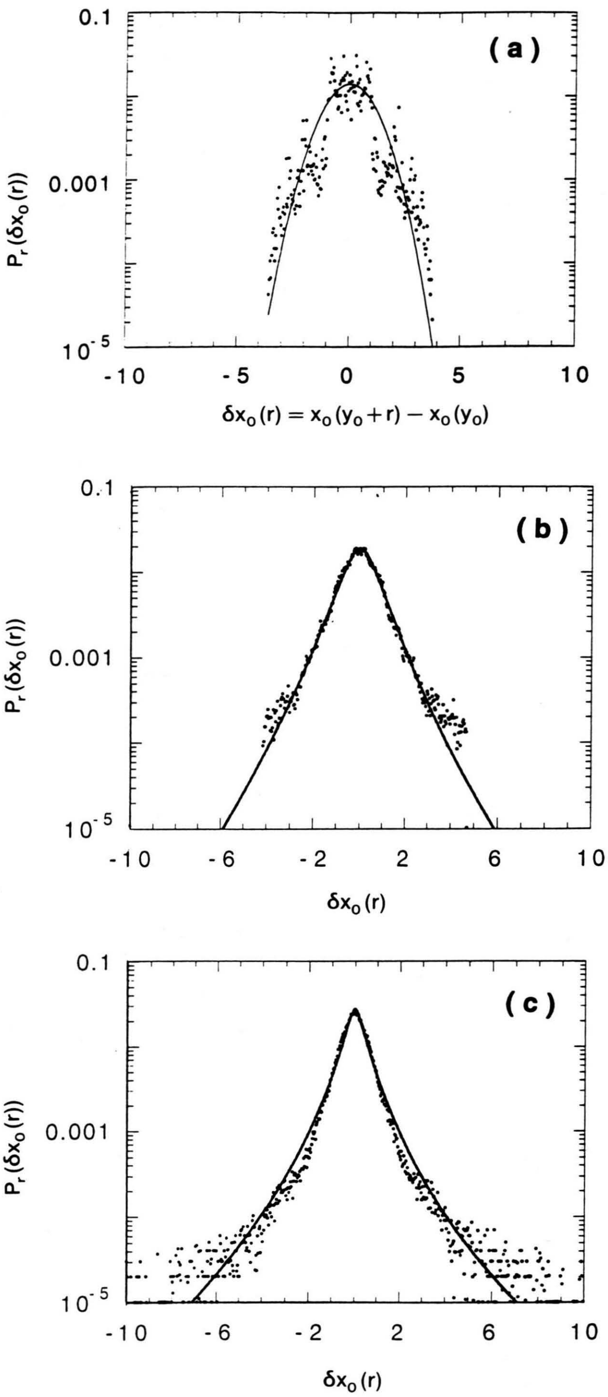

Fig. 5. Probability densities of $\delta x_0(r) = x_0(y_0+r) - x_0(y_0)$ for $r=0.16$ (a), $r=6.4 \cdot 10^{-4}$ (b) and $r=1.6 \cdot 10^{-4}$ (c). For comparison in (a) a Gaussian distribution and in (b) and (c) lognormal distributions are shown by solid lines. For the lognormal distributions the following form factors of (8) were chosen: $\lambda=0.48$ (b) and $\lambda=0.70$ (c) ($\delta x_0(r)$ is normalized to the standard deviation.

become more and more intermittent for small scales. As an interesting point it was found that only for r smaller than $3 \cdot 10^{-3}$ the phenomenon of lognormal shapes starts. Taking into account that the quantity $\delta\theta(r)$ should evaluate the typical fluctuations on the scale r, and that in our model the typical fluctuation on a given scale are given by the iteration step n, we can conclude that 8 folding and stretching processes are necessary to reach the intermittency region. This fact should be set in connection with the scaling behavior of the structure function (Figure 3). There a reasonable scaling is found for r values smaller than 10^{-2}.

The PDFs of the fractal curve are fitted with lognormal PDFs taken from [11]. These theoretical PDFs were evaluated for velocity difference in a turbulent flow. Here we used the symmetrical version

$$P(\delta u_r) \sim \int_0^{+\infty} (1/\sigma^2) \exp(-(\delta u_r)^2/2\sigma^2) \cdot \exp\{-(\ln\sigma)^2/2\lambda^2\} \, d\sigma, \qquad (8)$$

where λ is the form parameter. The larger λ is, the more lognormal the PDF will become. The best fits with this theoretical PDF are shown in Fig. 5 by solid lines. As it was also argued in [11], it is essential to investigate the r dependence of λ^2. For Kolmogorov's refined similarity hypotheses λ^2 should depend in a logarithmic way on r. In contrast to this, it is argued in [11] that for finite Reynolds numbers a power law dependence of λ^2 should be expected. For our fractal model a tendency to logarithmic scaling was found around $r=10^{-3}$, which may be the consequence of our fractal ansatz without any dissipative cutoff.

4. Discussions and Conclusions

A fractal curve was investigated with respect to characteristic features known from turbulence. The nonlinear scaling of the structure function exponents was found, as well as the changing shape of the PDFs as a function of the typical length scale r. After these results have been presented, we can now turn to the question of the reliability of the structure function exponents. In (1) the relation between the structure function and the PDF is shown. To evaluate $S^q(r)$, one has to perform an integration from minus infinity to plus infinity of a function $(\delta\theta(r))^q P_r(\delta\theta(r))$. For any numerical simulation or experiment this is impossible because $P_r(\delta\theta(r))$ is only known with finite precision.

The smallest probability is one over the total number of data. The point is that for any finite data the integrand of (1) should be known will some good precision around its maximum. The higher the factor q is, the larger $\delta\theta(r)$ becomes, where the maximum is located. Larger values of $\delta\theta(r)$ correspond to smaller probabilities. It is easy to verify that the more intermittent the PDF is, the more difficult it becomes to achieve a reasonable evaluation of (1) (see also the argumentation in [2]). Taking for example a PDF for $\lambda = 0.4$ (experimentally found [11]) more than 10^8 data points are necessary to evaluate $\zeta(10)$. Thus, we conclude that also in this paper the evaluation of $\zeta(q)$ is questionable for higher values of q. Nevertheless, it is easy to see that the result that $\zeta(q)$ displays a nonlinear behavior is not affected by this criticism. Note that an insufficient precision in the PDF causes an underestimation of the real value of the structure function. For decreasing r, i.e. for increasing intermittent PDFs, the underestimation becomes worse. For the scaling of the structure function with r as shown in Fig. 3, this has the consequence that there is in reality a smaller slope. If the scaling of (2) does not vanish, this has the consequence that the $\zeta(q)$ values are smaller for higher q values. For the q dependence of $\zeta(q)$ in Fig. 4a this leads to a more pronounced nonlinear behavior.

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